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Note

Structure of 4-connected claw-free graphs not containing a subdivision of K_5

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ABSTRACT

We characterize all the 4-connected claw-free graphs not containing a subdivision of K_5. We show that such graphs are either planar or can be constructed in a simple way starting from a triangle.

1. Introduction

All graphs in this paper are finite and simple. Paths and cycles are simple (that is, have no "repeated vertices") and are considered subgraphs.

For $X \subseteq V(G)$, the subgraph $G[X]$ induced on X is the subgraph with vertex set X and edge set all edges of G with both ends in X. ($V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively.) We say that $X \subseteq V(G)$ is a *claw in* G if $|X| = 4$ and $G[X]$ is isomorphic to the complete bipartite graph $K_{1,3}$. We say G is claw-free if no $X \subseteq V(G)$ is a claw in G.

If G is a graph and $A \subseteq G$, then $N_G(A)$ (or simply $N(A)$) denotes the set of vertices in $V(G) - V(A)$ adjacent to some vertex in $V(A)$. If $A = \{a\} \subseteq V(G)$, then we use $N_G(a)$ or $N(a)$ instead of $N_G(\{a\})$ or $N(\{a\})$.

If a path P has end vertices u and v, then P is called a u, v-path. For a graph G, and $A_i \subseteq V(G)$ or $A_i \subseteq G$ for $i = 1, 2$, we say P is an A_1, A_2-path if P is an a_1, a_2-path with $V(P) \cap V(A_i) = \{a_i\}$ for $i = 1, 2$. We say P is an A_1-path if P is an a_1, a_2-path with $V(P) \cap V(A_1) = \{a_1, a_2\}$.

We say that $S \subseteq V(G)$ *separates* A_1 from A_2 if every A_1, A_2-path contains a vertex from S. A *separation* of a graph G is a pair of subgraphs (G_1, G_2) of G, such that $E(G_1) \cap E(G_2) = \emptyset$, $G_1 \cup G_2 = G$, and $|V(G_1)|, |V(G_2)| > |V(G_1) \cap V(G_2)|$. The order of a separation (G_1, G_2) is $|V(G_1) \cap V(G_2)|$. A k-separation is a separation of order k. Observe that a graph G is k-connected if $|V(G)| > k$ and G has no k'-separation for any $k' < k$.

Let $k \geq 3$ be an integer and let $k_1 = \lceil k/2 \rceil$ and $k_2 = \lfloor k/2 \rfloor$. We say that a graph H is a *k-triangular-chain* if $V(H) = \{v_1, \ldots, v_{k_1}, u_1, \ldots, u_{k_2}\}$ and $E(H) = A_1 \cup A_2 \cup A_3 \cup A_4$, where $A_1 = \bigcup_{i=1}^{k_1-1} v_i v_{i+1}$, $A_2 = \bigcup_{i=1}^{k_2-1} u_i u_{i+1}$, $A_3 = \bigcup_{i=1}^{k_2} v_i u_i$, $A_4 = \bigcup_{i=1}^{k_1-1} u_i v_{i+1}$. See Fig. 1, for an 11-triangular-chain.

Let H be a $(2k-1)$-triangular-chain, where $k \geq 3$, and let $H_1 = H \cup \{v_k u_1, u_{k-1} v_1, v_1 v_k\}$. Then we say that H_1 is a $(2k-1)$-*strip*. See Fig. 2 for an example of an 11-strip. Note also that a 5-strip is isomorphic to K_5. The following is easy to observe.

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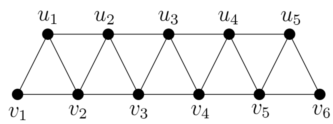

Fig. 1. An 11-triangular-chain.

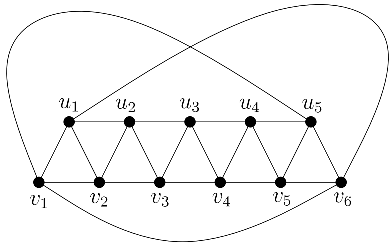

Fig. 2. An 11-strip.

Lemma 1.1. *Let H be a* $(2k-1)$*-strip, for some integer* $k \geq 4$*. Then H is claw-free, non-planar,* 4*-regular,* 4*-connected, and does not contain a subdivision of* K_5*.*

We can now state the main result of this paper.

1.2. *Let G be a* 4*-connected claw-free graph. Then exactly one of the following holds:*

1. *G contains a subdivision of* K_5*,*
2. *G is planar,*
3. *G is isomorphic to a* $(2k-1)$*-strip, for some* $k \geq 4$*.*

Recently, the first author has proved the following [3]. This was conjectured by Xiaoya Zha in 1992 [1].

1.3. *Let G be a* 4*-connected non-planar line graph. Then G contains a subdivision of* K_5*.*

We will show that 1.3 implies 1.2.

2. Proof of main result

A k-wheel (denoted by W_k) is a graph, H, with $V(H) = \{v_0, v_1, \ldots, v_k\}$ such that v_0 is adjacent to all vertices $v_1, \ldots, v_k$, and the vertices $v_1, \ldots, v_k$ induce a cycle. The vertex v_0 is called the *hub* of the wheel.

We need the following result regarding disjoint crossing paths [4].

2.1. *Let* $v_1, \ldots, v_k$ *be distinct vertices of a graph G. Then exactly one of the following holds.*

1. *there are disjoint paths of G,* P_1 *and* P_2*, so that* P_i *is a* p_i, q_i*-path,* $i = 1, 2$*, and* p_1, p_2, q_1, q_2 *occur in the sequence* $v_1, \ldots, v_k$ *in order (that is,* P_1 *and* P_2 *cross with respect to* $v_1, \ldots, v_k$*), or*
2. *there is a* (≤ 3) *separation* (G_1, G_2) *of G with* $v_1, \ldots, v_k \in V(G_1)$ *and* $|V(G_2 \setminus G_1)| \geq 2$*, or*
3. *G can be drawn in a disk with* $v_1, \ldots, v_k$ *on the boundary of the disk in order.*

We deduce the following.

2.2. *Let G be a* 4*-connected non-planar graph and suppose that G contains a subgraph,* $H \subseteq G$*, so that one of the following holds:*

1. *H is isomorphic to* K_4*, or*
2. *H is isomorphic to* W_k*, for some* $k \geq 4$*, and* $\deg_G(v_0) = k$ *(where* v_0 *is the hub of the wheel).*

Then G contains a subdivision of K_5*.*

Proof. We may assume that H is not isomorphic to K_4; for otherwise let $x \in V(G \setminus H)$ (x exists as G is non-planar). By Menger's Theorem, G has four x, H-paths, pairwise disjoint except for their end x. The union of these paths together with H is a subdivision of K_5 and the claim follows.

Hence assume that H is isomorphic to W_k, for some $k \geq 4$, and that $\deg_G(v_0) = k$. Let $V(H) = \{v_0, v_1, \ldots, v_k\}$, and assume that $v_1, \ldots, v_k$ appear on the cycle $H \setminus v_0$ in order. Let $G' = G \setminus v_0$. Clearly, G' cannot be drawn in a disk with $v_1, \ldots, v_k$ appearing on the boundary of the disk in order, for since $\deg_G(v_0) = k$ (and hence all neighbors of v_0 in G are in $V(H)$) that would imply that G is planar. Also, there is no (≤ 3)-separation (G'_1, G'_2) in G', so that $v_1, \ldots, v_k \in V(G_1)$ and $|V(G_2 \setminus G_1)| \geq 2$; for otherwise, all neighbors of v_0 in G are in $V(G'_1)$; but then $(G'_1 \cup v_0 \cup \{v_0v_1, \ldots, v_0v_k\}, G'_2)$ is a (≤ 3)-separation in G, contradicting the fact that G is 4-connected.

Hence, by 2.1, in G', there are disjoint paths P_1 and P_2, so that P_i is a p_i, q_i-path, $i = 1, 2$, and p_1, p_2, q_1, q_2 occur in the sequence $v_1, \ldots, v_k$ in order. It is easy to see (by taking minimal crossing subpaths of P_1 and P_2) that there are subpaths $P_1' \subseteq P_1$ and $P_2' \subseteq P_2$ of P_1 and P_2 (possibly $P_1' = P_1$ or $P_2' = P_2$), such that for $i = 1, 2$, P_i' has ends $p_i', q_i' \in \{v_1, \ldots, v_k\}$, P_i' is disjoint of $\{v_1, \ldots, v_k\}$ except for its ends, and p_1', p_2', q_1', q_2' appear on the sequence $v_1, \ldots, v_k$ in order. Then $H \cup P_1' \cup P_2' \subseteq G$ contains a subdivision of K_5 (with branch set $\{v_0, p_1', q_1', p_2', q_2'\}$) and the claim follows. □

The following result is straightforward. It follows from the fact that if G is a graph that is claw-free and triangle-free, then every vertex of G is of degree at most two.

2.3. *Let G be a connected graph. Then, G is claw-free and triangle-free if and only if G is a path or a cycle of length at least four.*

From 2.3 and 2.2, we obtain the following.

2.4. *Let G be a 4-connected claw-free non-planar graph. If G contains a vertex of degree at least five, then G contains a subdivision of K_5.*

Proof. Let $v \in V(G)$ be a vertex of G of degree at least 5. Let H be the subgraph induced by the neighbors of v. Clearly, H is claw-free (as $H \subseteq G$ is induced). Also, we may assume that H has no triangles, for otherwise the claim follows, as then G contains a subgraph isomorphic to K_4 and hence, by 2.2, a subdivision of K_5.

As $|V(H)| \geq 5$, H has no triangles, and $G[H \cup v] \subseteq G$ is claw-free, it must be that H is connected. To see this, assume that H is disconnected, and let C_1 be a connected component of H. Let $C_2 = H \setminus V(C_1)$. As $|V(H)| \geq 5$, there exists $1 \leq i \leq 2$, so that $|V(C_i)| \geq 3$. As H is triangle-free, there exist $a, b \in V(C_i)$ so that $ab \notin E(H)$ (and hence $ab \notin E(G)$). But then a, b and v together with any vertex of $V(C_{3-i})$ is a claw in G; a contradiction. Hence H is connected. We also see that H is not a path; for if H is a path, say $p_1, \ldots, p_{|V(H)|}$, then as $|V(H)| \geq 5$, $v, p_1, p_{\lceil |V(H)|/2 \rceil}, p_{|V(H)|}$ induce a claw in G; a contradiction.

2.3 implies then that H is a cycle. But then $G[V(H) \cup v]$ is isomorphic to $W_{\deg_G(v)}$, and the claim follows by 2.2. □

Proof of 1.2. Let G be a 4-connected claw-free graph. If G is planar or contains a subdivision of K_5, then there is nothing to prove. Hence, we assume that G is non-planar and does not contain a subdivision of K_5, and show that G is isomorphic to a k-strip, for some odd $k \geq 7$. By 2.4 and 2.2 we may assume the following.

(1) G is 4-regular and does not contain a subgraph isomorphic to K_4.

By 1.3, G is not a line graph. Hence, since G is claw-free but not a line graph, then by [2], G contains a subgraph, H, isomorphic to a 4-triangular-chain (i.e., isomorphic to the graph obtained from K_4 by deleting a single edge). Let H be such that

(i) H is isomorphic to a k-triangular-chain, for some $k \geq 4$;
(ii) subject to (i), k is maximal.

Let $k_1 = \lceil k/2 \rceil$ and $k_2 = \lfloor k/2 \rfloor$. We observe that:

(2) $V(H) = V(G)$.

Subproof. We may assume that k is even (for if k is odd, the proof follows the same arguments). Let $V(H) = \{v_1, \ldots, v_{k_1}, u_1, \ldots, u_{k_1}\}$ with edge set as in the definition of k-triangular-chain. Note that in H, v_{k_1} is of degree three. Let $x \in N_G(v_{k_1}) \setminus \{u_{k_1}, u_{k_1-1}, v_{k_1-1}\}$. By (ii), x is not adjacent to u_{k_1}. By (1), u_{k_1} is not adjacent to v_{k_1-1}. Hence, as G is claw-free, it must be that x is adjacent to v_{k_1-1}. But this implies that $k_1 = 2$ (for then v_{k_1-1} must be of degree strictly smaller than four in H), and using x we get a contradiction to (ii).

Hence, $N_G(v_{k_1}) \subseteq V(H)$ and by the argument above we may assume that $k \geq 6$. Then, $N_G(v_{k_1}) \cap \{v_1, u_1\} \neq \emptyset$. It follows that $V(G) = V(H)$, for if there exists $v \in V(G) \setminus V(H)$, then v_1, u_1, u_{k_1} is an ℓ-cut, $2 \leq \ell \leq 3$, separating v from v_2. This proves (2).

Hence assume that $V(G) = V(H)$. We see that k must be odd; for if k is even, then as G is 4-regular and simple it follows that $G = H \cup \{v_{k_1}v_1, u_1u_{k_1}, v_1u_{k_1}\}$, and then G is planar; a contradiction. Hence k is odd and $k \geq 5$, and we deduce as above that $G = H \cup \{v_1v_{k_1}, v_1u_{k_2}, u_2v_{k_1}\}$. Then G is isomorphic to a k-strip, for some $k \geq 5$. If $k = 5$, then G is isomorphic to K_5; a contradiction. Hence, $k \geq 7$, and the proof is completed. □

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